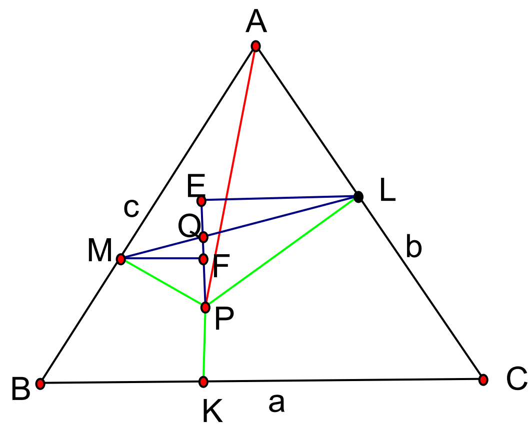
Basic inequality with distances.

(**Extraction from the notes on the theme "Inequalities with distances", Arkady Alt**)

It is well known inequality usually used as Lemma in the proof of Erdos-Mordell inequality, but important by itself, inequality

(**B**) $\boxed{aR_a \geq bd_c + cd_b.}$



Pic.E-M

Proof.

Let PK, PM, PL are perpendiculars from P to sides BC, CA, AB respectively. Then $d_a = |PK|$,

$d_b = PL,\ d_c := PM.$

Let LE and MQ be perpendiculars to $\overleftrightarrow{KP}$. Since $\angle MPF = \angle KBM = \angle ABC$ and $\angle LPE = \angle LCK = \angle ACB$ then $MF = d_c \sin\angle ABC$ and $LF = d_b \sin\angle ACB$ and we obtain $MF + LE \leq MQ + LQ = ML \iff d_c \sin\angle ABC + d_b \sin\angle ACB \leq ML.$

Since $\angle AMP = \angle ALP = 90^\circ$ then $R_a = AP$ is diameter of circumcircle for quadrilateral $ALPM$

then by sin-theorem $R_a = \dfrac{ML}{\sin\angle CAB} \iff ML = R_a \sin\angle CAB.$

Thus $R_a \sin\angle CAB \geq d_c \sin\angle ABC + d_b \sin\angle ACB$ and multiplying both sides of this inequality by

$2R,$ where R is circumradius of triangle $ABC,$ we finally obtain $aR_a \geq bd_c + cd_b.$

Equality condition in inequality (**B**) holds iff $EQ = 0$ and $FQ = 0,$ i.e. iff $ML \parallel BC.$